

RESTRICTION MAPS IN EQUIVARIANT KK -THEORY.

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ABSTRACT. We extend McClure's results on the restriction maps in equivariant K -theory to bivariant K -theory:

Let G be a compact Lie group and A and B be G - C^* -algebras. Suppose that $KK_n^H(A, B)$ is a finitely generated $R(G)$ -module for every $H \leq G$ closed and $n \in \mathbb{Z}$. Then, if $KK_*^F(A, B) = 0$ for all $F \leq G$ finite cyclic, then $KK_*^G(A, B) = 0$.

0. INTRODUCTION

Let G be a compact Lie group and let X be a finite G -CW-complex. For any closed subgroup $H \leq G$, we have a restriction functor in equivariant K -theory:

$$(0.1) \quad \text{res}_H^G : K_G(X) \rightarrow K_H(X).$$

As an application of the generalized Atiyah-Segal completion theorem of [AHJM88a], McClure proved the following.

Theorem (McClure [McC86, Theorem A]). *If $x \in K_G(X)$ restricts to zero in $K_H(X)$ for every finite subgroup H of G , then $x = 0$.*

Combining with Jackowski's result [Jac77, Corollary 4.3], one obtains the following.

Theorem (Jackowski-McClure [McC86, Corollary C]). *If $K_F^*(X) = 0$ for all $F \leq G$ finite cyclic, then $K_G^*(X) = 0$.*

We extend these to bivariant K -theory as follows.

Theorem 0.1. *Let G be a compact Lie group and A and B be G - C^* -algebras. Suppose that $KK_n^H(A, B)$ is a finitely generated $R(G)$ -module for every $H \leq G$ closed and $n \in \mathbb{Z}$.*

- (1) *Then, if $KK_*^F(A, B) = 0$ for all $F \leq G$ finite cyclic, then $KK_*^G(A, B) = 0$.*
- (2) *Suppose, in addition, that $KK_n^F(A, B)$ is a finitely generated group for all $F \leq G$ finite and $n \in \mathbb{Z}$. Then, if $x \in KK^G(A, B)$ restricts to zero in $KK^H(A, B)$ for all $H \leq G$ finite, then $x = 0$.*

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- Remark 0.2.** (1) See [MM04] for a dual result for restriction maps in K -homology of spaces with actions of discrete groups.
- (2) Theorem 0.1 is in stark contrast to the results of Heath Emerson, where he showed that even for circle actions, noncommutative algebras can behave very differently from commutative ones. [Eme10]

In fact, we prove the following. This is done mainly for clarity, but as an added bonus, we see that Theorem 0.1 holds for equivariant E -theory as well.

Theorem 0.3. *Let G be a compact Lie group and let \tilde{E}_G^* be an $RO(G)$ -gradable module theory over \tilde{K}_G^* . Suppose that $\tilde{E}_H^n(S^0)$ is a finitely generated $R(G)$ -module for every $H \leq G$ closed and $n \in \mathbb{Z}$. Let X be a finite based G -CW-complex.*

- (1) *Then, if $\tilde{E}_F^*(X) = 0$ for all $F \leq G$ finite cyclic, then $\tilde{E}_G^*(X) = 0$.*
- (2) *Suppose, in addition, that $\tilde{E}_F^n(S^0)$ is a finitely generated group for all $F \leq G$ finite and $n \in \mathbb{Z}$. Then, if $x \in \tilde{E}_G^*(X)$ restricts to zero in $\tilde{E}_H^*(X)$ for all $H \leq G$ finite, then $x = 0$.*

The proof follows [McC86] very closely. In Section 1, we show that Theorem 0.3 implies Theorem 0.1. In Section 2, we extend the generalized Atiyah-Segal completion theorem of [AHJM88a], supplying the missing ingredient needed to finish the proof in Section 3.

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1. $RO(G)$ -GRADED COHOMOLOGY THEORIES

Let G be a compact Lie group. A based G -space is a G -space with a G -fixed base point. In the rest of the paper, we assume that all G -spaces are G -CW-complexes and all cohomology theories are equivariant and reduced cohomology theories.

For a finite-dimensional representation V of G , we write S^V for the one-point compactification of V , considered a based G -space with base point the point at infinity.

1.1. $RO(G; U)$ -gradable theories. We fix a complete universe U . (cf. [May96, Definition IX.2.1]).

Definition 1.1. An $RO(G)$ -graded cohomology theory is an $RO(G; U)$ -graded cohomology theory in the sense of [May96, Definition XIII.1.1]. A \mathbb{Z} -graded cohomology theory is an $RO(G; U^G)$ -graded cohomology theory (any trivial universe would work). We say that a \mathbb{Z} -graded cohomology theory is $RO(G)$ -gradable if it is the \mathbb{Z} -graded part of an $RO(G)$ -graded theory.

Let \tilde{E}_G^* be a \mathbb{Z} -graded cohomology theory. For a closed subgroup $H \leq G$ and a based H -CW-complex X , we define

$$(1.1) \quad \tilde{E}_H^*(X) := \tilde{E}_G^*(G_+ \wedge_H X).$$

Then \tilde{E}_H^* is a \mathbb{Z} -graded cohomology theory on based H -spaces. If X is actually a based G -CW-complex, then we have a natural G -equivariant identification

$$(1.2) \quad G_+ \wedge_H X \cong G/H_+ \wedge X$$

and the collapse map $G/H \rightarrow *$ gives rise to a natural transformation

$$(1.3) \quad \text{res}_H^G : \tilde{E}_G^* \rightarrow \tilde{E}_H^*$$

called the *restriction* map.

1.2. Bivariant K -theory. The following is the main example we have in mind. First note that \tilde{K}_G^* is an $RO(G)$ -graded commutative ring theory with $\tilde{K}_G^V(X) = KK^G(C_0(S^V), C_0(X))$ and $R(G) \cong \tilde{K}_G(S^0)$.

Proposition 1.2. *Let G be a compact Lie group and let A and B be G - C^* -algebras. For a finite based G -CW-complex X and finite-dimensional real representation V of G , we define*

$$(1.4) \quad \tilde{E}_G^V(X) := KK^G(A \otimes C_0(S^V), B \otimes C_0(X)).$$

Then the following holds.

- (i) \tilde{E}_G^* defines an $RO(G)$ -graded cohomology theory on the category of finite based G -CW-complexes.
- (ii) \tilde{E}_G^* extends to an $RO(G)$ -graded cohomology theory on the category of based G -CW-complexes.
- (iii) \tilde{E}_G^* is a module theory over \tilde{K}_G^* .

Proof. (i) See [Kas88]. (ii) By Adams' representation theorem [May96, Theorem XIII.3.4], \tilde{E}_G^* is represented by an Ω - G -prespectrum, hence extends to an $RO(G)$ -graded cohomology theory on the category of G -CW-complexes. See [Sch92]. (iii) The module structure

$$(1.5) \quad \tilde{E}_G^V(X) \times \tilde{K}_G^W(Y) \rightarrow \tilde{E}_G^{V+W}(X \wedge Y).$$

is given by the Kasparov product

$$(1.6) \quad KK^G(A(S^V), B(X)) \times KK^G(C_0(S^W), C_0(Y))$$

$$(1.7) \quad \rightarrow KK^G(A(S^{V+W}), B(X \wedge Y)).$$

□

It is well-known that for $H \leq G$,

$$(1.8) \quad KK^G(A, B \otimes C_0(G/H_+)) \cong KK^H(A, B)$$

and the restriction map is induced by $G/H_+ \rightarrow S^0$. Hence we obtain the following corollary.

Corollary 1.3. *Suppose that Theorem 0.3 holds. Then Theorem 0.1 holds.* \square

2. ATIYAH-SEGAL COMPLETION

First we abstract the main finiteness condition from Theorem 0.3.

Definition 2.1. Let R be a unital commutative ring and let \tilde{E}_G^* be a \mathbb{Z} -graded cohomology theory with values in R -modules. We say that \tilde{E}_G^* is *finite* over R if $\tilde{E}_G^n(X)$ is a finitely generated R -module for every *finite* based G -CW-complex X and $n \in \mathbb{Z}$.

Clearly, this is equivalent to asking that $\tilde{E}_H^{k-n}(S^0) \cong \tilde{E}_G^k(G/H_+ \wedge S^n)$ is a finitely generated R -module for $H \leq G$.

Lemma 2.2. *Let G be a compact Lie group and let R be a unital commutative ring. Let \tilde{E}_G^* be a \mathbb{Z} -graded cohomology theory with values in R -modules. Suppose that R is Noetherian and \tilde{E}_G^* is finite over R . Then for any family \mathcal{I} of ideals in R , the following defines a \mathbb{Z} -graded cohomology theory with values in pro- R -modules:*

$$(2.1) \quad \tilde{E}_G^*(X)_{\mathcal{I}}^{\wedge} := \{\tilde{E}_G^*(Y)/J \cdot \tilde{E}_G^*(Y)\}.$$

where $Y \subseteq X$ runs over the finite based G -CW-subcomplexes of X and J runs over the finite products of ideals in \mathcal{I} .

Note that in this lemma, it is enough to have \tilde{E}_G^* to be a cohomology theory on *finite* based G -CW-complexes (only finite wedges are considered in the additivity axiom).

Proof. Exactness follows from the Artin-Rees lemma. See the proof of [AHJM88b, Lemma 2.1]. \square

2.1. Bott Periodicity. Let V be a complex G -representation. By Bott periodicity [Ati68, Theorem 4.3], $\tilde{K}_G^0(S^V)$ is a free $\tilde{K}_G^0(S^0)$ -module generated by the Bott element $\lambda_V \in \tilde{K}_G^0(S^V)$. The *Euler class* of V is defined to be $e^*(\lambda_V) \in \tilde{K}_G^0(S^0)$, where $e : S^0 \rightarrow S^V$ is the obvious map.

Lemma 2.3. *Let \tilde{E}_G^* be an $RO(G)$ -graded module theory over \tilde{K}_G^* . Then for any complex representation V , multiplication by the Bott element $\lambda_V \in \tilde{K}_G^0(S^V)$ gives an isomorphism*

$$(2.2) \quad \tilde{E}_G^0(S^0) \cong \tilde{E}_G^0(S^V).$$

If $V \subseteq W$ are complex representations and $i : S^V \rightarrow S^W$ is the inclusion, then the following diagram commutes

$$(2.3) \quad \begin{array}{ccc} \tilde{E}_G^0(S^0) & \longrightarrow & \tilde{E}_G^0(S^W) \\ \downarrow \chi_{W-V} & & \downarrow i^* \\ \tilde{E}_G^0(S^0) & \longrightarrow & \tilde{E}_G^0(S^V) \end{array}$$

Proof. Let $\lambda_V^{-1} \in \tilde{K}_G^V(S^0)$ denote the inverse Bott element: it has the property that

$$(2.4) \quad \lambda_V \cdot \lambda_V^{-1} = \lambda_V^{-1} \cdot \lambda_V = 1 \in \tilde{K}_G^V(S^V) \cong \tilde{K}_G^0(S^0).$$

Then multiplication by λ_V^{-1} gives the inverse map

$$(2.5) \quad \tilde{E}_G^0(S^V) \rightarrow \tilde{E}_G^V(S^V) \cong \tilde{E}_G^0(S^0).$$

The second statement is shown for $\tilde{E}_G^* = \tilde{K}_G^*$ in [AHJM88a, page 4]. The general case follows by functoriality. \square

2.2. Completion. A class of subgroups of G closed under subconjugacy is called a *family*. A family \mathcal{C} of subgroups of G determines a class, again denoted \mathcal{C} , of ideals of $R(G)$ by the kernels of the restriction maps:

$$(2.6) \quad \ker(\text{res}_H^G : R(G) \rightarrow R(H)), \quad H \in \mathcal{C},$$

hence a topology on any $R(G)$ -module.

The following is a straightforward generalization of [AHJM88a, Theorem 3.1].

Theorem 2.4. *Let G be a compact Lie group and let \tilde{E}_G^* be an $RO(G)$ -gradable module theory over \tilde{K}_G^* , which is finite over $R(G)$.*

Let \mathcal{C} be a family of subgroups of G . For any based G -CW-complex X , if $\tilde{E}_H^(X)_{\mathcal{C}|H}^\wedge = 0$ for all $H \in \mathcal{C}$, then $\tilde{E}_G^*(X)_{\mathcal{C}}^\wedge = 0$.*

Proof. By [Seg68, Corollary 3.3], $R(G) = \tilde{K}_G^0(S^0)$ is Noetherian. Hence, by Lemma 2.2, $\tilde{E}_G^*(X)_{\mathcal{C}}^\wedge$ is a cohomology theory.

Now the proof of [AHJM88a, Theorem 3.1] carries over ad verbatim, once we extend Bott periodicity to \tilde{E}_G^* as in Lemma 2.3. \square

Corollary 2.5. *Let EC denote the classifying space of \mathcal{C} . For any finite based G -CW-complex X , the projection map $EC_+ \rightarrow S^0$ gives completion*

$$(2.7) \quad \tilde{E}_G^*(EC_+ \wedge X) \cong \lim \tilde{E}_G^*(Y \wedge X) \cong \lim \tilde{E}_G^*(X)_{\mathcal{C}}^\wedge,$$

where Y runs over finite based subcomplexes of EC_+ .

Proof. The inverse system $\tilde{E}_G^*(X)_{\mathcal{C}}^\wedge$ satisfies the Mittag-Leffler condition and $\tilde{E}_G^*(Y \wedge X)$ is \mathcal{C} -complete for any finite based subcomplex $Y \subset EC_+$ (cf. [AHJM88a, Corollary 2.1]). \square

3. PROOF OF THEOREM 0.3

3.1. \mathcal{F} -spaces. Let \mathcal{F} be a family of subgroups of G . We say that a based G -CW-complex X is an \mathcal{F} -space if all the isotropy groups, except at the base point, are in \mathcal{F} . The following lemma says that in the proof of Theorem 0.3, we may assume that X is an \mathcal{F} -space, for any \mathcal{F} containing all finite cyclic subgroups of G .

Lemma 3.1. *Let G be a compact Lie group and let \tilde{E}_G^* be an $RO(G)$ -gradable module theory over \tilde{K}_G^* , which is finite over $R(G)$.*

Let \mathcal{F} be a family containing all finite cyclic subgroups of G . Then for any finite based G -CW-complex X , the top horizontal map in the commutative diagram

$$(3.1) \quad \begin{array}{ccc} \tilde{E}_G^*(X) & \longrightarrow & \lim_{Y \subset E\mathcal{F}_+} \tilde{E}_G^*(Y \wedge X) \\ \downarrow & & \downarrow \\ \prod_{F \in \mathcal{F}} \tilde{E}_F^*(X) & \longrightarrow & \lim_{Y \subset E\mathcal{F}_+} \prod_{F \in \mathcal{F}} \tilde{E}_F^*(Y \wedge X) \end{array},$$

is injective. Here Y runs over the finite based subcomplexes of $E\mathcal{F}_+$, the horizontal maps are induced by the projections $Y \wedge X \rightarrow X$ and the vertical maps are restrictions.

Proof. The \mathcal{F} -topology on $\tilde{E}_G^*(X)$ is Hausdorff by [McC86, Corollary 3.3]. Hence, the claim follows from Corollary 2.5. \square

Let \mathcal{C} denote the family of finite cyclic subgroups of G .

Proof of Theorem 0.3(1). By assumption, $\tilde{E}_F^*(X) = 0$ for all $F \in \mathcal{C}$. Let Y be a finite based G -CW-complex, which is a \mathcal{C} -space. Then the zero skeleton Y^0 and the skeletal quotients Y^n/Y^{n-1} are finite wedges of G -spaces of the form $G/F_+ \wedge S^n$ with $F \in \mathcal{C}$. It follows that $\tilde{E}_G^*(Y \wedge X) = 0$. Hence by Lemma 3.1, $\tilde{E}_G^*(X) = 0$. \square

3.2. Induction. We write \mathcal{O}_G for the category whose objects are orbit spaces G/H , where $H \leq G$ is a closed subgroup, and whose morphisms are homotopy classes of G -maps.

Recall that a compact Lie group is said to *cyclic* if it has a topological generator (an element whose powers are dense) and *hypercyclic* if it is an extension of a cyclic group by a finite p -group.

We write \mathcal{H} for the class of hypercyclic subgroups of G and let $\mathcal{O}_{\mathcal{H}}$ denote the full subcategory of \mathcal{O}_G of orbits G/H with H subconjugate to a subgroup in \mathcal{H} .

Lemma 3.2. *Let G be a compact Lie group and let \tilde{E}_G^* be an $RO(G)$ -gradable module theory over \tilde{K}_G^* . Then, for any based G -CW-complex, the restriction maps induce an isomorphism*

$$(3.2) \quad \tilde{E}_G^*(X) \cong \lim_{\mathcal{O}_{\mathcal{H}}} \tilde{E}_H^*(X).$$

Proof. Follows from Propositions 2.1 and 2.2 of [McC86]. \square

For any abelian group M , let $M_{\mathbb{Z}}^{\wedge}$ denote its adic completion $\lim_n M/nM$.

Proof of Theorem 0.3(2). Let \mathcal{F} denote the family of finite subgroups of G .

By Lemma 3.2, we may assume that G is a hypercyclic group and by Lemma 3.1, we may assume that X an \mathcal{F} -space.

Let G be a hyperelementary group and X an \mathcal{F} -space. Then the restriction map

$$(3.3) \quad \tilde{E}_G^*(X)_{\mathbb{Z}}^{\wedge} \rightarrow \lim_{F \in \mathcal{O}_{\mathcal{F}}} \tilde{E}_F^*(X)_{\mathbb{Z}}^{\wedge}.$$

is an isomorphism by [McC86, Theorem 1.1]. By [McC86, Corollary 3.3], the adic topologies on $\tilde{E}_G^*(X)$ and $\tilde{E}_F^*(X)$ are Hausdorff. This completes the proof. \square

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